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Stellar rotation and gravitational collapse: the *a/m* issue

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Abstract. We estimated that the ratio a/m between the specific angular momentum 'a' and the total mass 'm' (both geometrised units) of the cores of massive main-sequence stars exceeds the same ratio for neutron stars by about four orders of magnitude. We consider the question whether efficient mechanisms exist which damp away the excess amount of a/m during stellar evolution.

1. Introduction

The discovery of pulsars (Hewish *et al* 1968) and their interpretation as rotating magnetised neutron stars (Gold 1968) showed that catastrophic gravitational collapse occurs in nature as is predicted by the theory of relativity.

The rather small value of the critical mass for neutron star formation ($<3\,M_\odot$) inspires confidence that gravitational collapse to black holes is indeed possible whenever the masses involved exceed that upper limit.

General relativity states that a black hole is fully characterised by at most three parameters, namely total mass M, total angular momentum J, and total electric charge Q. These however must satisfy the relation (Misner *et al* 1973)

$$m^2 \geqslant a^2 + Q^2 \tag{1}$$

where $m = GM/c^2$, $a = J/Mc$ and Q are given in units of length, c is the velocity of light, G is the gravitational constant.

From (1) it follows that a necessary condition that a black hole existed as such, is that $a/m < 1$.

Neutron stars which are, after black holes, the most condensed equilibrium configurations in nature, have a very small ratio a/m, of the order of one hundredth, or much less, although they are believed to be originated by fast rotating main-sequence stars.

We have no direct observational evidence of collapsed configurations with a ratio $a/m > 1$, and this supports the conjecture that gravitational collapse beyond the neutron star phase can only lead to black hole formation. It is therefore compelling to clarify when and how the ratio a/m decreases during stellar evolution to the small amount which is now observed for the neutron stars. In this paper, we shall point

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out that the greater part of a/m might be lost by the neutron star after it has formed, supporting the possibility that a collapsed object with $a/m > 1$ can exist.

In § 2, we analyse the ratio a/m of a newly formed neutron star. Then we turn our attention to the evolution of the stellar cores before the neutron star formation, in the main sequence stage (§ 3), in the post main-sequence stage (§ 4) and collapsing stage, in order to get a self-consistent picture for the evolution of the ratio a/m. In § 5 we finally summarise the conclusions.

2. The ratio *a/m* for rotating neutron stars

Newtonian considerations ensure that the a/m for a rotating neutron star can be as high as one. At equilibrium, the radius of a neutron star reads

$$R_n = (3M/4\pi\rho_n)^{1/3} \tag{2}$$

where ρ_n is the star density which we assume uniform throughout. The critical radius for Newtonian stability, moreover, assuming spherical symmetry, rigid rotation and neglecting for simplicity the pressure gradient, reads

$$R_c = \tfrac{25}{4}(GM/c^2)(a/m)^2. \tag{3}$$

Imposing the condition $R_n > R_c$, we have

$$(a/m)^2(M/M_\odot)^{2/3} < (3/4\pi\rho_n M_\odot^2)^{1/3} 4c^2/25G \tag{4}$$

and with a standard value of $\rho_n = 5 \times 10^{14}$ g cm^{-3}, equation (4) leads to the following inequality for the neutron star formation:

$$a/m < 1.1(M/M_\odot)^{-1/3}. \tag{5}$$

A one-solar mass neutron star can tolerate a ratio a/m up to or slightly larger than one; however no neutron star seems to exist with such a high value of a/m.

The observational evidence of neutron stars in fact was possible after pulsars were discovered and it was found that they rotate with a very short period of about 1 s.

Calling P the rotational period of a neutron star and assuming that it rotates rigidly with uniform density throughout, the ratio a/m reads in this case

$$(a/m)_{NS} = (4\pi c/5G)R^2/MP. \tag{6}$$

Using in (6) the standard values for a neutron star, i.e. $R \sim 10^6$ cm, $M = 1M_\odot$, we find that even for the fastest rotating neutron star associated with the Crab pulsar with $P = 33 \times 10^{-3}$ s, we have $(a/m)_{NS} \sim 1.7 \times 10^{-2}$, much less than one.

Let us then consider what the period P^* would be, should the ratio $(a/m)_{NS}$ be ≥ 1. With the same standard parameters as before, we have

$$\text{period} \leq P^* = 0.57 \times 10^{-3}\ \text{s}.$$

All the known pulsars have a period much larger than P^*, so all pulsars have a ratio $a/m \ll 1$.

The neutron stars' rotational period however changes because of two main processes which take place at the expense of their rotational energy: (i) the continued radiation which originates from them due to the existence of an off-centred dipole magnetic field; (ii) the impulsive radiation reaction force which accelerates a new-born neutron star to a high translational velocity (Harrison and Tademaru 1975). Process (ii) works mainly at the early stages of a pulsar formation while process (i) works at all times after a pulsar is formed; in the latter case the rate of change of the period, $\dot{P}$, is known for many pulsars with a good amount of precision.

Let us consider process (i). In order to know the initial period P_0 at the end of the acceleration phase since the neutron star's birth from the values of P and $\dot{P}$, we need to know the pulsar age. This is precisely known only for the Crab pulsar, $t_{\mathrm{Crab}} = 928$ years. Using the formula

$$t = \tfrac{1}{2}(P/\dot{P})(1 - P_0^2/P^2) \tag{7}$$

we deduce $P_0 = 17 \times 10^{-3}$ s for the Crab, which is still larger than P^*. For all the other pulsars for which P and $\dot{P}$ are known, one can calculate the expected age t^* they would have, had their initial period P_0 been equal to P^*. Since however $P^* \ll P$ for all the other pulsars, equation (7) reduces to the standard expected age $t^* = P/2\dot{P}$, which is generally believed to be larger than the true age (Manchester and Taylor 1977), so we can conclude that all pulsars have at the end of their acceleration period a ratio $a/m \ll 1$.

Most important is the process (ii) which implies that a large amount of rotational energy is lost since the very formation of the neutron star, in favour of a large translational velocity. This effect is believed to take place because of some inherent anisotropy in the supernova outburst, which then leads to the emission of asymmetric electromagnetic radiation. Pulsars in fact appear to be high velocity objects with a translational velocity larger than $100\ \mathrm{km\,s^{-1}}$ (Trimble 1971, Galt and Lyne 1972, Ewing *et al* 1970). A detailed analysis of the selection effects on the scintillation observations of the pulsar radiation leads to a most probable lower limit for pulsar velocities of $120\ \mathrm{km\,s^{-1}}$ (Hanson 1979, Lyne 1980).

Following Harrison and Tademaru (1975), we have

$$M\,\mathrm{d}V/\mathrm{d}t = -(\varepsilon/c)I\Omega\,\mathrm{d}\Omega/\mathrm{d}t \tag{8}$$

where V is the translational velocity, ε is the radiation asymmetry coefficient, Ω is the angular velocity of the neutron star and I is its moment of inertia.

From (8) it follows that

$$MV_0 = (\varepsilon/2c)I(\Omega_i^2 - \Omega_0^2) \tag{9}$$

where Ω_i is the initial angular velocity of a neutron star and Ω_0 is that at the end of the accelerating period; similarly V_0 is the final translational velocity at the end of that period. With an $\Omega_0 < 3.7 \times 10^2\ \mathrm{s^{-1}}$ (corresponding to $P_0 > 1.7 \times 10^{-2}$ s), $V \sim 120\ \mathrm{km\,s^{-1}}$, $I \sim 10^{45}\ \mathrm{g\,cm^2}$, we have, with a modest $\varepsilon \sim 10^{-2}$,

$$\Omega_i \approx \Omega^* = 1.28 \times 10^4\ \mathrm{s^{-1}}. \tag{10}$$

To this value would then correspond an initial $P_i \approx 0.49 \times 10^{-3}\ \mathrm{s} \approx P^*$ which implies from (6) an initial $(a/m)_{\mathrm{NS}} \approx 1$. The consistency of this result with (5) inspires further confidence in the Harrison and Tademaru mechanism; it would also suggest that the occurrence of gravitational collapse of stellar cores at the end of stellar evolution with values of the ratio a/m of the order of one and even larger than one is indeed possible.

3. The ratio *a/m* for the stellar cores of the main sequence stars

It has long been realised that the ratio a/m for normal stars of early spectral type is much larger than one. In fact, observational evidence of mass loss in Be stars suggests that these stars have critical rotational velocities for equatorial shedding; in this case the virial theorem ensures that the ratio τ between the rotational kinetic energy and the total gravitational energy ranges between 0.14 and 0.25 (Ostriker and Tassoul 1969). Moreover the relation between a/m and τ simply reads

$$a/m = 1.2c\tau/V_e$$

so for an observed rotational velocity $V_e = 400\ \mathrm{km\ s^{-1}}$ (Slettebak 1970) and $\tau = (0.14\text{–}0.25)$, the ratio ranges as $a/m = (126\text{–}225)$. Although it is evident from the observations that evolving stars lose mass and angular momentum, it is not obvious what the behaviour of the ratio a/m will be.

The angular momentum distribution within a star is largely unknown and the same is true for the degree of rotational distortion, so we shall consider idealised cases only.

Case A: Rigid body rotation

Within the assumption of a uniform angular velocity distribution, we calculated the ratio a/m for a number of main sequence stars with masses ranging from 1 to 10 $M_\odot$; we also assumed them to be spherical with uniform (mean) density $\bar{\rho}$ throughout. Table 1 shows these values together with the relevant parameters of the stars; the

Table 1. a/m for main sequence stars.

M ($M_\odot$)	R (10^{10} cm)	$V_e \sin i$ (km $\mathrm{s^{-1}}$)	a (km)	m (km)	a/m
1.0	7.0	10	9.3	1.5	6.2
1.1	7.8	29	30	1.7	17.7
1.2	8.3	90	100	1.8	55.6
1.3	8.9	126	149	2.0	74.5
1.4	9.5	148	187	2.1	89.0
1.5	10.0	161	215	2.3	93.5
1.6	10.5	175	245	2.4	102
1.7	11.0	183	269	2.6	103
1.8	11.4	191	291	2.7	108
1.9	11.9	197	313	2.9	108
2.1	12.3	204	335	3.2	105
2.2	13.2	210	370	3.3	112
2.3	13.7	231	422	3.5	121
2.5	14.5	270	522	3.8	137
2.7	15.3	280	571	4.1	139
3.0	16.5	306	673	4.5	150
3.3	17.6	320	751	5.0	150
3.8	19.2	333	853	5.7	150
4.3	20.6	365	1003	6.5	154
4.8	22.0	377	1106	7.2	154
5.5	23.6	387	1218	8.3	147
6.3	25.4	395	1338	9.5	141
7.7	28.8	395	1517	11.6	131
10	32.1	400	1713	14.0	122

angular momentum per unit mass 'a' is defined as

$$a = 0.4RV_e/c \tag{11}$$

where R is the radius of the star adapted from Tinsley (1980) and V_e is the equatorial rotational velocity which we approximated with the average apparent velocities $\langle V_e \sin i\rangle$ listed in column 3 (Rajamohan 1978, Bernacca and Perinotto 1970).

Theoretical models predict that these stars will eventually undergo core collapse after a sequence of exothermic reactions to iron-peak elements. This stage is believed to lead to type II supernova explosions. What would then be the ratio a/m for such a core prior to catastrophic collapse to a neutron star or possibly black holes? Let us consider a $1M_\odot$ stellar core of a main sequence star with the following parameters (see table 1):

$$M = 10\,M_\odot, \qquad R = 3.3\times10^{11}\ \mathrm{cm}, \qquad V_e = (2\text{–}4)\times10^{7}\ \mathrm{cm\ s^{-1}},$$
$$\Omega = V_e/R = (0.6\text{–}1.2)\times10^{-4}\ \mathrm{s^{-1}}. \tag{12}$$

With the assumption of rigid rotation, the ratio a/m for the stellar core will depend on the density distribution. For simplicity, assuming that the density of the core is uniform, the expression of a/m for the core reads

$$a/m = (2c/5GM)R^2\Omega = 5.5\times10^5(M/M_\odot)^{-1/3}\rho^{-2/3}\Omega. \tag{13}$$

Let us then consider a $1\,M_\odot$ core of this $10\,M_\odot$ star. The average density $\bar{\rho}$ of the whole star is 0.15 g cm^{-3} and the central density ρ_c is about 7.8 g cm^{-3} (Schwarzschild 1965). Taking several typical values for the core density, the corresponding values for a/m of the core are shown in table 2. There the most plausible estimation of the ratio a/m for the $1\,M_\odot$ core within the assumption of rigid rotation is in the range $a/m = (10–50)$.

Table 2. a/m for $1\,M_\odot$ core of a $10\,M_\odot$ main sequence star.

ρ_{core} (g cm^{-3})	Ω (s^{-1})	a/m
0.15	$(0.6–1.2)\times10^{-4}$	117–234
3	$(0.6–1.2)\times10^{-4}$	16–32
7.8	$(0.6–1.2)\times10^{-4}$	8.5–17

Case B: Differential rotation

If convective motions are important in the stellar interior, then it is most likely that the angular momentum is redistributed so that differential rotation sets up. However, the angular momentum distribution in the rotating star is quite unknown. Bodenheimer (1971) has built rotating main-sequence stellar models by assigning various angular momentum distribution laws. Let us pick up some data from his models and evaluate a/m for their corresponding $1\,M_\odot$ cores. Qualitatively, in all those cases, the stellar cores have more angular momentum than they would have in the corresponding rigid rotating case, i.e. a/m will be larger.

In table 3, five models are listed for our purpose; in the last line of table 3, a/m is evaluated from (13) by assuming that the core has a uniform density $\rho = \rho_c$ and uniform angular velocity $\Omega = \Omega_c$, where ρ_c and Ω_c are the central density and the central angular velocity of the corresponding model.

Model I is a rigid rotating 15 $M_\odot$ star with a similar V_e as discussed in case A. The resulting a/m is also similar. Model II describes a differentially rotating star with the same J and M as in model I; the ratio a/m is therefore larger and the resulting V_e is smaller. Since V_e of model II seems too small compared with the typical observed values, we choose another 15 $M_\odot$ model with the same angular momentum distribution law as in model II, but larger J. That is model III. Correspondingly the ratio a/m is about 10^2, which is larger than the values estimated in case A. Models IV and V describe a 30 $M_\odot$ star with two different angular momentum distribution laws and reasonable values of V_e. In both cases, the ratio a/m is of the same order of 10^2.

Combining cases A and B, we can conclude that a/m for a 1 $M_\odot$ stellar core of massive rotating main-sequence stars with V_e equal to (200–400) km s^{-1} ranges from 10 to 100.

4. Evolution of the ratio a/m during the post main-sequence stage

The post main-sequence evolution of a star is characterised by a strong contraction of its helium rich core and a corresponding expansion of the surrounding envelope.

During the core contraction and envelope expansion, the angular momentum distribution must change due to angular momentum transfer from core to envelope. Endal and Sofia (1977, 1978) computed the post main-sequence evolution of a rotating star with a realistic time-dependent redistribution of angular momentum. They found that convection and Eddington circulation are the most important mechanisms for changing the angular momentum distribution. For our purpose, the point is to know how much angular momentum can be transported out from the core during the whole post main-sequence stage. Let us see what information we can get from their computational data.

They evolved a 7 $M_\odot$ stellar model from zero age main sequence with rigid rotation at a rate $\Omega = 8.8 \times 10^{-5}$ s^{-1} to the double shell burning period. Their final configuration has a central density $\rho_c = 1.7 \times 10^7$ g cm^{-3} and $\Omega_{\text{core}} \approx 0.4$ s^{-1} (from their figure 4, $\log \Omega_{\text{core}} \approx -0.4$). The core density of a 7 $M_\odot$ main sequence star should be assumed to be 10 g cm^{-3}; from equation (13), a/m of the core is evaluated as 10.4. Instead, assuming an average core density $\bar{\rho} = 1 \times 10^7$ g cm^{-3}, a/m of the final configuration is evaluated as 5. That is to say, half of the initial angular momentum has been carried out during this period.

They have also worked out a 10 $M_\odot$ stellar model and evolved it from zero age main sequence to carbon ignition, a time span covering about 98% of the time to core collapse. They found that the effect of the gas-dynamical redistribution is to decrease the total angular momentum of the core by 40%.

The Endal and Sofia models seem to say that the pre-collapsing core will still have a value of a/m larger or much larger than one (say, in the range (5–50). However, the effect of a magnetic field has not been included in their models. Although Endal and Sofia did not think it is important, there is reason to believe that it can be far more effective than the material interchange (Mestel 1981).

Table 3. a/m for 1 $M_\odot$ core of a main sequence star in differentially rotating model. CGS units are used.

	I	II	III	IV	V
Angular momentum distribution law	$\Omega = \text{constant}$	$j(\mu) = \frac{5J}{2M}[1-(1-\mu)^{2/3}]$	same as II	same as II	$j(\mu) = \frac{J}{M} 2\mu$
Total angular momentum	3.17×10^{52}	3.17×10^{52}	1.12×10^{53}	5.7×10^{53}	5.01×10^{53}
Total mass	$15\, M_\odot$	$15\, M_\odot$	$15\, M_\odot$	$30\, M_\odot$	$30\, M_\odot$
Equatorial velocity V_e	4.12×10^{7}	0.75×10^{7}	2.56×10^{7}	4.22×10^{7}	3.47×10^{7}
Equatorial radius R_e	4.10×10^{11}	3.56×10^{11}	3.73×10^{11}	5.74×10^{11}	4.90×10^{11}
Equatorial angular velocity Ω_e	1.0×10^{-4}	2.1×10^{-5}	6.9×10^{-5}	7.4×10^{-5}	7.1×10^{-5}
Central angular velocity Ω_c	1.0×10^{-4}	1.7×10^{-4}	5.8×10^{-4}	5.8×10^{-4}	5.9×10^{-4}
Central density ρ_c	5.2	5.2	6.8	4.7	4.2
Re/Rp	1.12	1.06	1.76	2.69	1.41
a/m	18.3	31.2	88.9	114	125

5. Conclusions

A $1\,M_\odot$ core in the main sequence star with mass in the range (10–30) $M_\odot$ and equatorial velocity in the range (200–400) $\mathrm{km\,s^{-1}}$ should have the value of a/m in the range (10–100). The observed rotating neutron stars, on the other hand, have the value of a/m less or much less than 10^{-2}. The question arises therefore of when and how the core reduces its a/m of about four orders of magnitude during stellar evolution (table 4). The main conclusions of our paper can be summarised as follows.

Table 4. Evolution of a/m.

Period	Range of a/m or main mechanisms for a/m losing
Main sequence	$a/m = (10-100)$
Post main sequence	Convection and circulation
	Magnetic field
Collapse	Neutrino emission
	Gravitational radiation
Neutron star evolution	Harrison–Tademaru mechanism
	Pulsar radiation
Observed neutron star	$a/m \lesssim 10^{-2}$

(1) If the Tademaru–Harrison mechanism is actually taking place at the early stage of neutron star formation, a zero age neutron star should have the value of a/m nearly equal to one. Consequently, the reduction of a/m from (10–100) to 1 should take place in the post main-sequence stage and the collapsing stage.

(2) If the Endal and Sofia models did in fact consider the dominant mechanisms for reducing a/m during the post main-sequence stage, the pre-collapsing cores should have a/m in the range (5–50). Then we ask if there is any mechanism which reduces a/m from (5–50) to 1 in the dynamical timescale of gravitational collapse. As estimated (Endal and Sofia 1977, Kazanas 1977), neutrino emission will carry away less than 10% of the total angular momentum. Then what about gravitational radiation? Preliminary calculations (de Felice *et al* 1982) seem to show that it is also not efficient enough. We suspect that the magnetic field is probably playing an important role during the post main-sequence evolution, so detailed calculation should be done in this sense (Mestel 1981, Wilson 1978).

(3) If the Tademaru–Harrison mechanism does not take place, a/m should be reduced from (5–50) to 10^{-2} in the dynamical timescale. Our feeling is that this is impossible. In this case, a more realistic post main-sequence stellar model is crucial to clarify the evolution of the ratio a/m. It may give a constraint such that any rotating post main-sequence stellar model which cannot reduce a/m by more than one order of magnitude is not a realistic one.

Finally, what about the very situation in which the gravitational collapse does not give rise to a neutron star? In this case ($M_{\mathrm{core}} > 3\,M_\odot$), should we expect the gravitational collapse always to lead to a black hole state? If case (i) holds, then the possibility of a continued gravitational collapse with a/m larger than one is plausible and that poses serious questions whether nature necessarily recognises that the ratio a/m should become less than one to form an event horizon, or it may allow avoidance of black hole formation.

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